Tensor and Coupled Decompositions in Block Terms: Uniqueness and Irreducibility

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***Abstract*—In this work, we present recent results concerning decompositions of tensors and ensembles of matrices in sum of terms that are not necessarily rank-1. We formulate mathematically the concept of irreducibility, which is the enabling factor that allows these low-rank terms to exist as "blocks" without being further factorized into terms of smaller rank. We first demonstrate these results on tensors. Next, we generalize our results to a coupled factorization of several matrices that cannot be written as a single tensor. This coupled factorization is inspired by data fusion, and generalizes independent component analysis in several directions.**

Outline

Decomposition of a tensor in a sum of rank-1 terms is a basic and prevalent tool in data analysis. However, in various scenarios, writing a tensor as a sum of terms whose rank is other than one is more faithful to the underlying structure of the data (see, e.g., [1] and references therein). Such models are based on the observation that in real-world data, the assumption that each latent phenomena can be described by rank-1 terms is sometimes too restrictive. For this aim, different types of tensor block term decomposition (BTD) [2] have been proposed. When applicable, these models provide advantages such as higher precision and computational efficiency over their rank-1 counterparts (e.g., [3], [4]).

A decomposition of a tensor in a sum of rank-1 terms means that we write it as a sum of rank-1 tensors. A rank-1 tensor of order N (an N-dimensional "cube") is an outer product of N rank-1 vectors. A rank-1 tensor is the most minimal form to write a tensor of order N, in the sense that it can be represented by only N vectors: we cannot write an Nth-order tensor as an outer product of less than N vectors (some or all of which might be identical).

This observation seems trivial. But, what happens when we speak of representing a tensor as a sum of terms that are not rank-1 tensors? How come such decompositions exist? Why don't we write each of these terms as a sum of several rank-1 tensors? How can these low-rank terms (which are not rank-1 tensors) have a meaning on their own? This work is motivated by these questions.

Fundamental concepts in decompositions in a sum of terms of rank other than one are *reducibility* and *irreducibility*. In this context, irreducibility means that each term in the sum cannot be further factorized into several distinct terms of smaller rank by linear transformations, that is, by multiplying each mode of the tensor by an invertible matrix. Irreducibility is irrelevant to decompositions in sum of rank-1 terms, because each mode already has the smallest possible rank which is one.

We say "in this context", because the term "(ir)reducibility" is used in algebra also in other meanings. Our use of "(ir)reducibility" is strongly related to that used in representation theory [5]. In fact, it was shown [6], [7], [8] that the irreducibility, and thus also the uniqueness and identifiability, of joint block diagonalization (JBD) (illustrated in Fig. 1a), and of the rank-$(L_r, M_r, \cdot)$ BTD (illustrated in Fig. 1b), can be characterized using Schur's lemma [5], when the other factor matrices are nonsingular. In this work, we shall explain how this result leads to a possible new generalization of the concept of Kruskal rank [9]. Kruskal's rank is a key concept in defining the uniqueness of the decomposition of a tensor in sum of rank-1 terms.

Earlier work on the uniqueness of BTD focused on role of the factor matrices (e.g., the matrix $\mathbf{A}$ and its transpose in Fig. 1), but not on the role of the values on the block-diagonals of the tensor $\mathcal{S}$ (see Fig. 1). In order to avoid reducibility, it was assumed that the block-diagonal terms, also known as *core tensors* [2], were generic. Given this assumption, it was proven in earlier work that BTD were generically unique, under mild conditions on the factor matrices.

In this work, we thus enrich our understanding of block decompositions by taking into account non-generic core tensors [8]. Using this approach, we prove that BTD (specifically, a type-2 decomposition of a tensor in a sum of rank-$(L_r, M_r, \cdot)$ terms, illustrated in Fig. 1b) may be non-unique when the core tensors are irreducible yet not generic. This observation is not covered by other existing results in the literature. We show that if the diagonal blocks (on the frontal slice of the tensor) have different size, they cannot cause non-uniqueness; this raises the question whether the set of block dimensions in a BTD can be regarded as a type of diversity [10]. Next, we show how to extend our analytical framework, that is based on Schur's lemma, to the an ensemble of matrices that cannot be stacked in a single tensor [11]. We describe several ways in which this new *coupled block decomposition/diagonalization* (illustrated in Fig. 2) subsumes known results on the rank-$(L_r, M_r, \cdot)$ BTD. We obtain a conjecture on a new generalization of Kruskal's rank for the core tensors. We explain how our results provide new insights on the uniqueness of canonical polyadic decomposition (CPD) and BTD.

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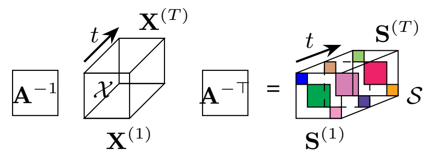
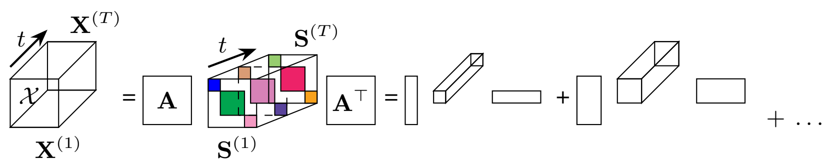


(a) Joint block diagonalization (JBD) can be written in matrix notation, $\mathbf{A}^{-1}\mathbf{X}^{(t)}\mathbf{A}^{-\top} = \mathbf{S}^{(t)}$ as the multiplication of each matrix $\mathbf{X}^{(t)}$ by $\mathbf{A}^{-1}$ and its transpose. Each matrix $\mathbf{S}^{(t)}$, $1 \leq t \leq T$, is block diagonal with the same block pattern for all t. This is a joint congruence transformation that yields a set of block-diagonal matrices, hence the name JBD. Instead, we can write this in tensor notation: $\mathcal{X} \times_1 \mathbf{A}^{-1} \times_2 \mathbf{A}^{-1} = \mathcal{S}$, where $\times_n$ in the nth-mode tensor product.



(b) Decomposition of a 3rd-order tensor $\mathcal{X}$ in a sum of block terms. In this example, there are three blocks on each diagonal, hence, three block terms; only the first two are depicted explicitly on the right hand side. The 3rd-order tensor $\mathcal{X}$ can be written as a multiplication of a tensor $\mathcal{S}$, whose frontal slices are block-diagonal matrices with the same block pattern, with matrix $\mathbf{A}$ and its transpose, on the first and second mode. Note that here, there is no multiplication on the third mode. The illustrated decomposition is a special case of the $(L_r, M_r, \cdot)$-BTD [2].

Fig. 1: Illustration of a decomposition of a 3rd-order tensor into block terms. When the transformation matrix $\mathbf{A}$ is invertible, the only non-uniqueness of this decomposition may occur due to pathological values in the diagonal blocks [8].

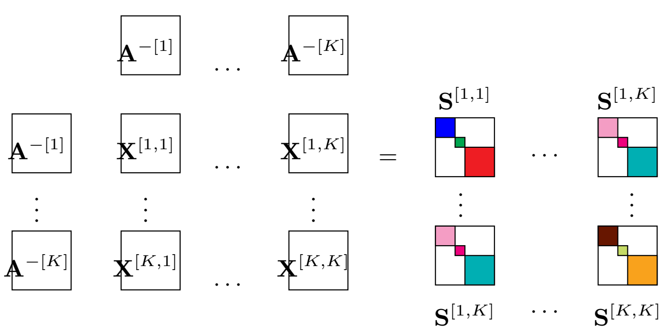


Fig. 2: Illustration of coupled block diagonalization (CBD). In this example, we have an ensemble of matrices, $\{\mathbf{X}^{[k,\ell]}\}$, indexed by $1 \leq k, \ell \leq K$, where $K \geq 2$, and K transformations (invertible matrices) $\mathbf{A}^{[k]}$. Each matrix $\mathbf{X}^{[k,\ell]}$ is multiplied by two transformation matrices: $\mathbf{A}^{-[k]}\mathbf{X}^{[k,\ell]}\mathbf{A}^{-[\ell]\top} = \mathbf{S}^{[k,l]}$. Each matrix $\mathbf{S}^{[k,l]}$ is block diagonal: in this example, it has three rectangular (in this case, they are illustrated as square, but this does not have to be the case in general) non-overlapping matrices on its main diagonal, and the off-diagonal terms are zero. This type of decomposition, which was introduced in [12], is more general than JBD, because we can obtain JBD as a special case of CBD, by (i) setting $\mathbf{A}^{[k]} = \mathbf{A}$ for all k, and (ii) $\mathbf{S}^{[k,\ell]}$ all have the same size and the same block-diagonal structure (these requirements are already satisfied in the illustration, but this does not have the be the case in general). Since CBD is more general than JBD, its uniqueness cannot be determined using Schur's lemma [5]. Therefore, we use new variants of Schur's lemma [11] to characterize its uniqueness. It turns out that for this decomposition, non-uniqueness may occur for pathological cases in the values of the diagonal blocks [13], but not only: it may also depend on the sizes of the blocks [14].

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